CS103 Handout 38
Spring 2017 May 21, 2017

Extra Practice Problems 8

We've gotten a number of requests for more practice problems involving the pigeonhole principle, so here's an entire packet of practice problems to work through that focuses purely on pigeonhole-type results. Some of these problems directly reference the pigeonhole principle, while others are simply based on the idea of counting up how many objects of some type there are.

Hope this helps!

Forced Connectivity

Let $G = (V, E)$ be a graph with n nodes. Prove that if the degree of every node in V is at least $(n-1) / 2$, then G is connected.

Lattice Points

A ***lattice point*** in 2D space is a point whose (x, y) coordinates are integers. For example, (137, -42) is a lattice point, but $(1.5, \pi)$ isn't.

Suppose that you pick any five lattice points in 2D space. Prove that there must be some pair of points in the group with the following property: the midpoint of the line connecting those points is also a lattice point.

Planar Graphs

Recall from lecture that a ***planar graph*** is a graph that can be drawn in the 2D plane such that no two edges cross one another. The ***four-color theorem*** says that all planar graphs are 4-colorable.

A ***k-clique*** is a graph consisting of k nodes that are all adjacent to one another. Prove that the 5-clique is not planar.

Finite Sets

(Midterm exam, Fall 2013)

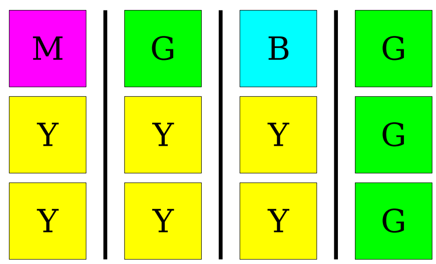
Let $f : A \to A$ be a function from a finite set A to itself. (A finite set is one whose cardinality is a natural number). Prove that if f is injective, then f is surjective.

Colored Cubes*

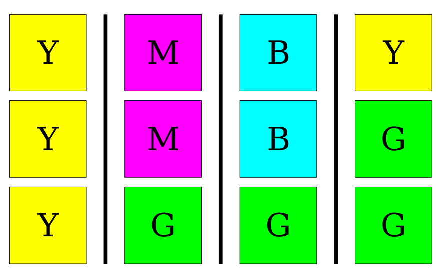
Suppose that you have a collection cubes of n different colors. For simplicity we'll assume that the total number of cubes you have is a multiple of n; specifically, let's suppose that you have kn total cubes, where k is some natural number. For example, you might have 30 cubes of six different colors, in which case $n = 6$ and $k = 5$. Alternatively, you might have 200 cubes of 40 different colors, where $n = 40$ and $k = 5$.

Now, let's suppose that you have n bins into which you can place the cubes, each of which holds exactly k different cubes. Although it may not seem like it, it's always possible to distribute the cubes into the boxes such that every box is full (that is, it has exactly k cubes in it) and that each box has cubes of at most two different colors. Prove this fact using induction on n, the number of colors.

Some examples might help here. Suppose that $n = 4$ and $k = 3$, meaning that there are four different colors of cubes, twelve total cubes, and four boxes that hold three cubes each. The goal is then to put the cubes into the four boxes such that every box has exactly three cubes and contains cubes of at most two different colors. If you have six yellow (Y) cubes, four green (G) cubes, one blue (B) cube, and one magenta (M) cube, here's one way to distribute them:



If you have four yellow (Y) cubes, four green (G) cubes, two blue (B) cubes, and two magenta (M) cubes, you could distribute them this way:



The result you're proving in this problem forms the basis for the *alias method*, a fast algorithm for simulating rolls of a loaded die. This has applications in machine learning (simulating different outcomes of a random event), operating systems (allocating CPU time to processes with different needs), and computational linguistics (generating random sentences based on differently-weighted rules).

* This problem adapted from Exercise 3.4.1.7 of *The Art of Computer Programming, Third Edition, Volume II: Seminumerical Algorithms* by Donald Knuth.